Thermal stress analysis of hollow cylinder

November 17, 2017

Analytical solution

Consider a hollow infinite cylinder with inner radius R_1 and outer radius R_2. Under the axisymmetric condition, we consider a stationary temperature field described by the following equation

$$\frac{1}{r}\frac{d}{dr}\left(\kappa r\frac{dT}{dr}\right) = 0 \tag{1}$$

and the boundary conditions

$$-\kappa\frac{dT}{dr}|_{R_1} = q_b, \quad T|_{R_2} = T_b \tag{2}$$

with κ the thermal conductivity. We assume that κ is constant and the influx of the heat power is positive by denoting $q = -q_b$, we can derive the analytical solution to the axisymmetry thermal problem

$$T(r) = \frac{R_1 q}{\kappa}\ln\left(\frac{R_2}{r}\right) + T_b \tag{3}$$

with T the temperature increment from the initial state, which is zero in this problem.

Under such temperature field, the induced stresses of the cylinder are given by

$$\begin{aligned}
\sigma_r &= \psi\frac{du_r}{dr} + \lambda\frac{u_r}{r} - \beta T \\
\sigma_\theta &= \lambda\frac{du_r}{dr} + \psi\frac{u_r}{r} - \beta T \\
\sigma_z &= \lambda\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) - \beta T
\end{aligned} \tag{4}$$

where $\psi = \lambda + 2\mu$, $\beta = \alpha(3\lambda + 2\mu)$ with λ, the Lamé elastic constant, μ the shear modulus, and α, the line thermal expansion coefficient.

By omitting the source term, the momentum balance equation takes the form

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \tag{5}$$

Substituting the stress expressions (4) into the momentum balance equation (5) leads to an equation of radial displacement as

$$\frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) - \frac{\beta}{\psi}\frac{dT}{dr} = 0 \tag{6}$$

Substituting the temperature solution (3) into eqn (6) gives

$$\frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) = -\frac{qR_1\beta}{\psi\kappa}\frac{1}{r} \tag{7}$$

Integrating eqn (7) twice yields the analytic solution of radial displacement induced by the temperature field described by eqn. (1) and (2):

$$u_r = -\frac{qR_1\beta}{2\psi\kappa}r\left(\ln r - \frac{1}{2}\right) + \frac{A_0}{2}r + \frac{A_1}{r} \tag{8}$$

where A_0 and A_1 are coefficients that can be determined by the boundary conditions. Furthermore, the radial stress can be derived as

$$\begin{aligned}\sigma_r =&\psi\left[-\frac{qR_1\beta}{2\psi\kappa}\left(\ln r+\frac{1}{2}\right)+\frac{A_0}{2}-\frac{A_1}{r^2}\right]\\&+\lambda\left[-\frac{qR_1\beta}{2\psi\kappa}\left(\ln r-\frac{1}{2}\right)+\frac{A_0}{2}+\frac{A_1}{r^2}\right]\\&-\beta\left[\frac{R_1 q}{\kappa}\ln\left(\frac{R_2}{r}\right)+T_b\right]\end{aligned} \tag{9}$$

We assume the cylinder surface are traction free, that is

$$\sigma_r|_{R_1}=0, \quad \sigma_r|_{R_2}=0, \tag{10}$$

To obtain the coefficients, we rewrite eqn (9) into the following form

$$\begin{aligned}\sigma_r =&-\frac{qR_1\beta}{2\psi\kappa}(\lambda+\psi)\ln r+\frac{qR_1\beta}{4\psi\kappa}(\lambda-\psi)\\&+\frac{A_0}{2}(\lambda+\psi)+(\lambda-\psi)\frac{A_1}{r^2}\\&-\beta\left[\frac{R_1 q}{\kappa}\ln\left(\frac{R_2}{r}\right)+T_b\right]\end{aligned} \tag{11}$$

Applying the boundary conditions to eqn (11) yields the coefficients as

$$A_0 = 2\left[\frac{qR_1\beta}{2\psi\kappa}\left(\frac{\lambda-\psi}{\lambda+\psi}\right)\left(-\frac{1}{2}+\frac{R_1^2}{R_2^2-R_1^2}\ln\frac{R_2}{R_1}\right)+\frac{qR_1\beta}{2\psi\kappa}\ln R_2+\frac{\beta T_b}{\lambda+\psi}\right] \tag{12}$$

$$A_1 = -\frac{qR_1\beta}{2\psi\kappa}\frac{R_1^2R_2^2}{R_2^2-R_1^2}\ln\frac{R_2}{R_1} \tag{13}$$

Eventually, the radial stress is obtained as

$$\sigma_r = \frac{qR_1\beta}{2\psi\kappa}(\lambda-\psi)\left[\ln\frac{R_2}{r}+\frac{R_1^2}{R_2^2-R_1^2}\left(1-\frac{R_2^2}{r^2}\right)\ln\frac{R_2}{R_1}\right] \tag{14}$$

It is interesting to point out that the outer surface temperature does not has any influence to the radial stress under the traction free boundary condition.

GS/RF results and comparision

The problem is solved by using GeoSys/Rockflow and the analytical solution with constants $T_b = 25°\text{C}$, $q = 30W/m^2$, $R1 = 4.5m$ and $R_2 = 50m$ and material parameters given in Table 1

Table 1: Material properties for TM coupled axisymmetrical problem

Property	Value	Unit
Young's modulus	2.5×10^3	MPa
Poisson's ratio	0.25	$--$
Thermal expansion	4.25×10^{-5}	$--$
Thermal conductivity	5.5	$W/m°\text{C}$

Fig. 1, 2 and 3 plot the obtained variables along the radial direction. The numerical results agree well with the analytical ones.

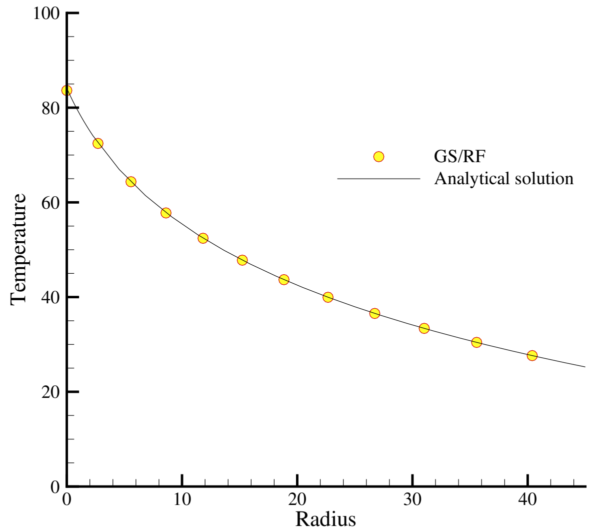

Figure 1: Profile of temperature

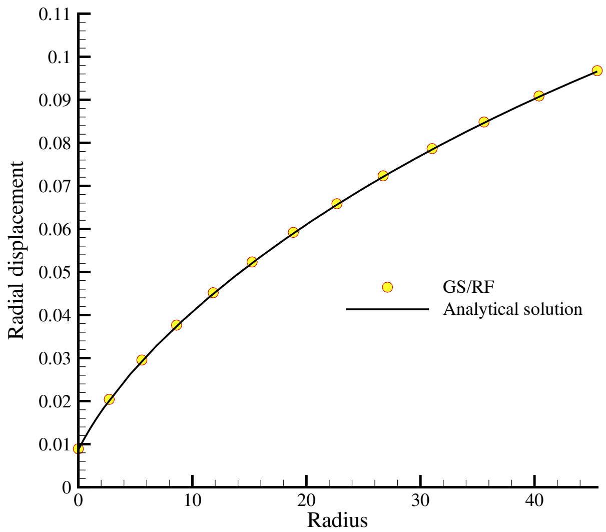

Figure 2: Profile of the radial displacement

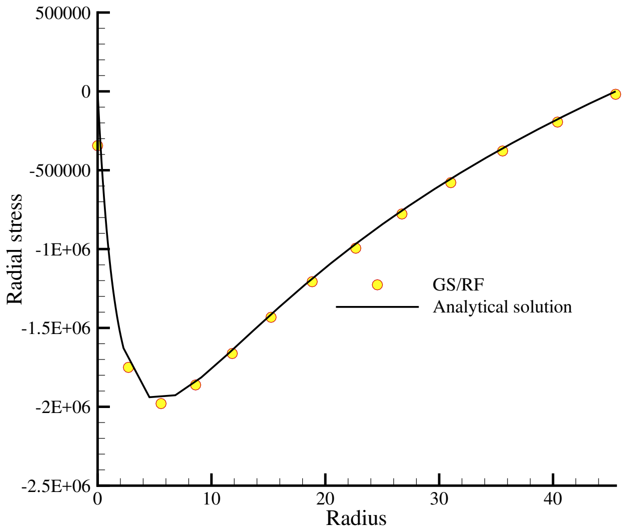

Figure 3: Profile of the radial stress